

LANDSTAD-VAES THEORY FOR LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. Landstad-Vaes theory concerns the structure of the crossed product of a C^* -algebra by an action of locally compact (quantum) group. In particular it describes the position of original algebra inside crossed product. The problem was solved in 1979 by Landstad for locally compact groups and in 2005 by Vaes for regular locally compact quantum groups. To extend the result to non-regular groups we modify the notion of G -dynamical system introducing the concept of weak action of quantum groups on C^* -algebras. It is still possible to define crossed product (by weak action) and characterise the position of original algebra inside the crossed product. The unpleasant feature of our theory is potential non-uniqueness of crossed product. At the end we discuss a few examples.

0. INTRODUCTION

The concept of crossed product of a C^* -algebra by an action of a locally compact (quantum) group comes from the desire to unite in a single object the C^* -algebra and the unitaries implementing the action. For the theory of crossed product of C^* -algebras by actions of locally compact groups see [13].

One of the most used formula in the operator algebra theory is the implementing of an automorphism by a unitary operator:

$$(0.1) \quad \alpha(d) = UdU^*.$$

In this formula U is a unitary operator acting on a Hilbert space K and d runs over a C^* -algebra D of operators acting on K . It is assumed that $\alpha(d) \in D$ for any $d \in D$ and that $\alpha(D) = D$. Then α is an automorphism of C^* -algebra D . We say that automorphism α is implemented by U .

Any automorphism of a C^* -algebra D can be implemented. For any α there exist a Hilbert space K and a pair (j, U) , where j is an embedding of D in $B(K)$ and U is a unitary operator acting on K such that (identifying d with $j(d)$) we have (0.1). We say that (j, U) is a covariant representation of D .

It is interesting to extend D by including U . Let

$$(0.2) \quad B = \left\{ U^n d : n \in \mathbb{Z}, d \in D \right\}^{\text{CLS}},$$

where CLS stays for norm closed linear span. Then B is a C^* -algebra, $U \in M(B)$, $D \subset B$ and $DB = B$. The latter means that the embedding $D \subset B$ is a morphism from D into B . In general, for given D and α , the algebra B may depend on the used covariant representation. For instance, if α is inner then we may take $U \in M(D)$ and then $B = D$.

To obtain a more interesting algebra B we have to assume that U is in a sense independent of elements of D . One of the symptom of this independence is the existence of dual action. We say that B admits a dual action if for any $z \in S^1$ there exists an automorphism β_z of B such that $\beta_z(U) = zU$ and $\beta_z(d) = d$ for any $d \in D$. It turns out that in any case one can find a covariant representation such that the algebra (0.6) admits a dual action. Moreover the algebra B with the dual action is unique (up to isomorphism): It does not depend on the choice of covariant representation. This unique C^* -algebra

2000 *Mathematics Subject Classification.* 46L55 (46L08 81R50).

Key words and phrases. Dynamical system, Crossed product, Landstad conditions, Landstad algebra, Non-regular quantum groups.

S. Roy was supported by a Fields–Ontario postdoctoral fellowship and the NSERC grant of Matthias Neufng. S.L. Woronowicz was supported by the Alexander von Humboldt-Stiftung and by the National Science Center (NCN) grant no. 2015/17/B/ST1/00085. Also the authors are very grateful to the Oberwolfach Research Institute for Mathematics for creating the perfect conditions for fruitful work within the program "research in pairs" in August 2014.

is denoted by $D \rtimes_{\alpha} \mathbb{Z}$ and called the crossed product of D by the automorphism α . It is equipped with the dual action β of S^1 and distinguished element $U \in M(D \rtimes_{\alpha} \mathbb{Z})$ such that

$$(0.3) \quad \beta_z(U) = zU$$

for any $z \in S^1$.

Conversely assume that B is a C^* -algebra equipped with an action β of S^1 and a distinguished element $U \in M(B)$ such that formula (0.3) holds. Then

$$D = \{d \in B : \beta_z(d) = d \text{ for all } z \in S^1\}$$

is a C^* -subalgebra of B , $DB = B$, formula (0.1) defines an automorphism α of D and relation (0.2) holds.

In the above the single automorphism may be replaced by a locally compact quantum group of automorphisms. We shall consider locally compact quantum group $G = (A, \Delta)$. In this case the formula (0.1) takes the form

$$(0.4) \quad \alpha(d) = U(d \otimes I)U^*,$$

where $U \in M(B_0(K) \otimes A)$ is a unitary representation of G acting on K . It is assumed that $\alpha(d) \in M(D \otimes A)$ for any $d \in D$ and that (using the notation (1.1))

$$(0.5) \quad \alpha(D)(I_D \otimes A) = D \otimes A.$$

Then $\alpha \in \text{Mor}(D, D \otimes A)$. We say that α is an action of G on D implemented by a representation U .

In general, a right action of G on a C^* -algebra D is a faithful morphism $\alpha \in \text{Mor}(D, D \otimes A)$ such that $(\alpha \otimes \text{id}) \circ \alpha = (\text{id} \otimes \Delta) \circ \alpha$. The action is said to be continuous if the Podleś condition (0.5) holds. In the similar way one defines left actions.

Any continuous action α of a locally compact group G on a C^* -algebra D can be implemented. For any α there exist a Hilbert space K and a pair (j, U) , where j is an embedding of D in $B(K)$ and U is a unitary representation of G acting on K such that (identifying d with $j(d)$) we have (0.4). We say that (j, U) is a covariant representation of (D, α) . Moreover one may assume that U is weakly contained in the regular representation. It means that U is of the form

$$U = (\psi \otimes \text{id})V,$$

where $\psi \in \text{Rep}(\widehat{A}, K)$ and $V \in M(\widehat{A} \otimes A)$ is the canonical bicharacter establishing duality between $G = (A, \Delta)$ and $\widehat{G} = (\widehat{A}, \widehat{\Delta})$. In what follows we shall use shorthand¹ notation $V_{\psi 2} = (\psi \otimes \text{id})V$.

Again it is interesting to extend D by including ψ -copy of \widehat{A} . We shall use notation (1.1) (see next section). Let

$$(0.6) \quad \begin{aligned} B &= \left\{ \psi(\widehat{a})d : \widehat{a} \in \widehat{A}, d \in D \right\}^{\text{CLS}} \\ &= \psi(\widehat{A})D. \end{aligned}$$

Then B is a C^* -algebra, $D, \psi(\widehat{A}) \subset M(B)$ and $DB = \psi(\widehat{A})B = B$. The latter means that the embedding $D \subset M(B)$ is a morphism from D into B and ψ is a morphism from \widehat{A} into B . We refer to the next section for the concept of morphism in the category of C^* -algebras. In general, for given D and α , the algebra B may depend on the used covariant representation.

To obtain a more interesting algebra B we have to assume that elements of \widehat{A} are in a sense independent of elements of D . One of the symptom of this independence is the existence of dual action. We say that B admits a dual action if there exists a faithful morphism $\beta \in \text{Mor}(B, \widehat{A} \otimes B)$ such that $\beta(\psi(\widehat{a})) = (\text{id} \otimes \psi)\widehat{\Delta}(\widehat{a})$ and $\beta(d) = I \otimes d$ for any $d \in D$ and $\widehat{a} \in \widehat{A}$. It turns out that in any case one can find a covariant representation such that the algebra (0.6) admits a dual action. Moreover the algebra B with the dual action is unique (up to isomorphism): It does not depend on the choice of covariant representation. This unique C^* -algebra is denoted by $D \rtimes_{\alpha} G$ and called the crossed product of D by the action α . It is equipped with the dual action β of \widehat{G} and the morphism $\psi \in \text{Mor}(\widehat{A}, B)$ such that

$$(0.7) \quad \beta(\psi(\widehat{a})) = (\text{id} \otimes \psi)\widehat{\Delta}(\widehat{a})$$

¹This is extended leg numbering notation

for any $\widehat{a} \in \widehat{A}$. In general, a triple (B, β, ψ) , where B is a C^* -algebra, $\beta \in \text{Mor}(B, \widehat{A} \otimes B)$ is a left action of \widehat{G} on B and ψ is a faithful morphism from \widehat{A} into B is called G -product if formula (0.7) holds.

One of the aim of the Landstad theory was to describe position of D within $M(D \rtimes_\alpha G)$. It could be easily shown that elements $d \in D$ satisfy the following three conditions:

- L1. $\beta(d) = I_{\widehat{A}} \otimes d$,
- L2. $\psi(\widehat{a})d \in B$ for any $\widehat{a} \in \widehat{A}$,
- L3. $(V_{\psi 2}(d \otimes I_A)V_{\psi 2}^*)(I_B \otimes a) \in M(B) \otimes A$ for any $a \in A$.

The above conditions were (in a more classical language) formulated by Magnus Landstad² (see also section 7.8 of [13]). Assuming that G is a locally compact group Landstad was able to show that his conditions completely characterise those elements of $M(B)$ that belong to D . More than that Landstad proved (for classical G) that any G -product (B, β, ψ) comes from crossed product construction:

$$B = D \rtimes_\alpha G,$$

where D is the C^* -algebra consisting of all elements of $d \in M(B)$ satisfying conditions L1, L2 and L3, and

$$(0.8) \quad \alpha(d) = V_{\psi 2}(d \otimes I_A)V_{\psi 2}^*$$

for any $d \in D$.

Instead of looking for conditions characterising elements of D within $M(D \rtimes_\alpha G)$ one may formulate properties concerning the C^* -algebra D itself. It could be easily shown that

- V1. $\beta(d) = I_{\widehat{A}} \otimes d$ for any $d \in D$,
- V2. $B = \psi(\widehat{A})D$,
- V3. $(V_{\psi 2}(D \otimes I_A)V_{\psi 2}^*)(I_B \otimes A) = D \otimes A$.

These conditions were formulated by Stefaan Vaes in [18]. Assuming that G is a locally compact regular quantum group Vaes was able to show that for any G -product (B, β, ψ) there exists unique C^* -subalgebra D of $M(B)$ satisfying conditions V1, V2 and V3. This subalgebra is equipped with the left action $\alpha \in \text{Mor}(D, A \otimes D)$ of G introduced by (0.8) and (B, β, ψ) comes from crossed product construction: $B = D \rtimes_\alpha G$. This way Vaes extended Landstad theory to regular locally compact quantum groups.

It is interesting to compare Landstad and Vaes conditions. Let (B, β, ψ) be a G -product, $D \subset M(B)$ be a C^* -subalgebra and $d \in D$. Then L1 coincides with V1, L2 follows from V2 and assuming V3 we see that for any $a \in A$ we have $(V_{\psi 2}(d \otimes I_A)V_{\psi 2}^*)(I_B \otimes a) \in D \otimes A \subset M(B) \otimes A$. It shows that L3 follows from V3.

Let (B, β, ψ) be a G -product. In general (for non-regular G) a subalgebra $D \subset M(B)$ satisfying Vaes conditions may not exist. To regain the existence of D we have to replace V3 by a weaker condition C3. Roughly speaking, C3 means that the slices of $V_{\psi 2}(D \otimes I_A)V_{\psi 2}^*$ generate C^* -algebra D .

In what follows the subalgebra $G \subset M(B)$ satisfying conditions V1, V2 and C3 will be called Landstad algebra of (B, β, ψ) . We shall prove that any G -product admits unique Landstad algebra. Unfortunately now (when condition V3 is not satisfied) formula (0.8) does not define a action of G on D . In general $\alpha(d) \notin M(D \otimes A)$. To deal with the problem we invent the notion of weak action adapted to this situation. In brief instead of (0.5) we assume that slices of $\alpha(d)$ belongs to D and that the set of all slices generate C^* -algebra D . In the following a pair (D, α) , where D is a C^* -algebra and α is a weak action of G on D , will be called a G -dynamical system. For regular groups the concepts of weak and continuous actions coincides.

Working with weak actions we have to reconsider the concept of crossed product. Given a G -dynamical system (D, α) , we shall construct a G -product (B, β, ψ) such that D plays the role of Landstad algebra of (B, β, ψ) , α is implemented by $V_{\psi 2}$ and $\psi \in \text{Mor}(\widehat{A}, B)$ is the canonical embedding. In other words B is in a sense crossed product of D by the action α and β is the dual action. However

²see formulae (3.6) - (3.8) in [9]. In fact in the second condition Landstad demanded additionally that $d\psi(\widehat{a}) \in B$. However this requirement is redundant, it follows from other Landstad conditions (see last section)

there is some price to pay at this point: we are not able to show that (B, β, ψ) is unique. It means that according to our present knowledge the correspondence between G -products and G -dynamical systems could be many to one.

Let us shortly discuss the content of the paper. In section 1 we explain the notation used in the paper. In particular we recall the category of C^* -algebras (concepts of morphisms and composition of morphisms). We also collect all informations concerning quantum groups used in the paper. Section 2 contains main definitions and results. We introduce (recall) the concept of G -dynamical system and G -product and describe the duality between them. In section 3 we analyse the concept of weak action. At the end we show that for regular groups any weak action is continuous (satisfies Podleś condition). Section 4 is devoted to the crossed product construction. We start with a G -dynamical system (D, α) . As always the crossed product is of the form XY , where $X = \widehat{\Delta}(\widehat{A})$ is a copy of \widehat{A} and $Y = \widetilde{\alpha}(D)$ is a copy of D . The key problem is to show that XY is a C^* -algebra. We shall prove a technical result (Proposition 4.1) that is well adapted to such problems. The existence and uniqueness of Landstad algebra for any G -product is discussed in section 5. At the end of the section we show the uniqueness of G -product corresponding to any G -dynamical system (D, α) with weakly continuous action α . Next, in section 6 we present a number of examples of G -products with not obvious Landstad algebras. In the last section we show (for coameanable G) that one of the original Landstad condition is a consequence of the others.

1. NOTATION

Throughout the paper we shall use the following notation: For any separable Hilbert space K we set

$B(K)$ = the von Neumann algebra of all bounded operators acting on K ,

$B_0(K)$ = the C^* -algebra of all compact operators acting on K ,

$B(K)_*$ = the set of all normal functionals on $B(K)$
= the set of all continuous functionals on $B_0(K)$,

$$C^*(K) = \left\{ A \subset B(K) : \begin{array}{l} A \text{ is separable } C^* \text{-algebra} \\ \text{such that } AK = K \end{array} \right\}.$$

Then $B(K)$ and $B_0(K)$ are C^* -algebras, $C^*(K)$ is a set of C^* -algebras, $B_0(K) \in C^*(K)$. In this paper phrase ‘ C^* -algebra generated by a set’ means ‘the smallest C^* -algebra containing the set’.

Let X and Y be norm closed subsets of a C^* -algebra. We set

$$(1.1) \quad XY = \left\{ xy : \begin{array}{l} x \in X \\ y \in Y \end{array} \right\}^{\text{CLS}},$$

where CLS stays for norm closed linear span.

For any C^* -algebra A , $M(A)$ will denote the multiplier algebra (cf [13]) of A . Then A is an essential ideal in $M(A)$. We shall use the category of C^* -algebras introduced in [20, 19]. It will be denoted by C^* . Objects are C^* -algebras. For any C^* -algebras A and B , $\text{Mor}(A, B)$ is the set of all $*$ -algebra homomorphisms φ from A into $M(B)$ such that $\varphi(A)B = B$. Any $\varphi \in \text{Mor}(A, B)$ admits unique extension to a unital $*$ -homomorphism $\widetilde{\varphi} : M(A) \rightarrow M(B)$. If $\varphi \in \text{Mor}(A, B)$ and $\psi \in \text{Mor}(B, C)$ (A, B, C are C^* -algebras) then the composition of morphisms $\psi \circ \varphi \in \text{Mor}(A, C)$ is defined as composition of $*$ -algebra homomorphisms: $\psi \circ \varphi = \psi \circ \widetilde{\varphi}$.

Let A be a C^* -algebra. Depending on the context we shall use two symbols: id_A and I_A to denote the identity map acting on A . id_A will denote the identity morphism acting on A , whereas I_A will denote the unit element of the multiplier algebra $M(A)$ ³. To simplify notation we write I_K and id_K instead of $I_{B_0(K)}$ and $\text{id}_{B_0(K)}$. We shall often omit the index ‘ A ’ when the algebra A is obviously implied by the context.

Let $X, Y \in C^*(K)$, where K is a Hilbert space. Assume that

$$XY = YX.$$

³Identifying elements of $M(A)$ with left multipliers acting on A we have $I_A = \text{id}_A$.

Then $Z = XY \in C^*(K)$, $X, Y \subset M(Z)$ and the embeddings are morphisms from X and Y into Z . We shall refer to this situation by saying that Z is a crossed product of X and Y .

The category C^* admits the natural monoidal structure given by tensor product functor. In this paper we use exclusively minimal (spacial) tensor product of C^* -algebras.

We shall use the following shorthand notation invented in [3]: If $(X_\omega)_{\omega \in \Omega}$ is a family of subsets of a Banach space X , then the smallest Banach subspace of X containing all X_ω ($\omega \in \Omega$) will be denoted by

$$\left[X_\omega : \omega \in \Omega \right].$$

So we have:

$$\left[X_\omega : \omega \in \Omega \right] = \left(\bigcup_{\omega \in \Omega} X_\omega \right)^{\text{CLS}}.$$

Typically we shall deal with expressions of the form

$$\left[(\omega \otimes \text{id})Z : \omega \in B(H)_* \right],$$

where Z is a linear subset of $B(H \otimes K)$ and H and K are Hilbert spaces. Let $A \in C^*(H)$. By the factorisation theorem [4] any $\omega \in B(H)_*$ is of the form $\omega' a$ and of the form $a \omega'$ where $\omega' \in B(H)_*$ and $a \in A$. Therefore

$$\begin{aligned} (1.2) \quad \left[(\omega \otimes \text{id})Z : \omega \in B(H)_* \right] &= \left[(\omega \otimes \text{id})(A \otimes I)Z : \omega \in B(H)_* \right] \\ &= \left[(\omega \otimes \text{id})Z(A \otimes I) : \omega \in B(H)_* \right]. \end{aligned}$$

We shall refer to this formula by saying that ω *emits* A to the right (upper formula) or to the left (lower formula). Reading from the right we say that ω *absorbs* A .

In the present paper we consider actions of a locally compact quantum group on C^* -algebras. The group will be denoted by G . We shall not use the full power of the theory developed by Kustermans and Vaes in [8] (see also [10]). Instead we shall assume that G is constructed from a manageable (modular) multiplicative unitary V in the way described in [24] (see also [15, 16]). In particular we do not assume the existence of Haar measures.

The Haar measure plays an essential role in the Landstad considerations. To construct β invariant elements in $M(B)$ he applies β_g ($g \in G$) to specially chosen elements of B and then integrates over G using the Haar measure. On the other hand in Vaes approach [18] (at least in the part that intersects with our interest) the Haar measure plays purely decorative role and can be removed from considerations. Then the proofs become more transparent.

To fix the notation we quote below the basic formulae from [2] and [24]. Let H be a Hilbert space and \overline{H} be the Hilbert space complex-conjugate to H . Then we have canonical anti-unitary mapping $H \ni x \mapsto \overline{x} \in \overline{H}$ and transposition map $B(H) \ni m \mapsto m^\top \in B(\overline{H})$ introduced by the formula

$$m^\top \overline{x} = \overline{m^* x}$$

for any $x \in H$.

We shall use leg numbering notation. Throughout the paper we fix unitary operators $V \in B(H \otimes H)$ and $\tilde{V} \in B(\overline{H} \otimes H)$ and a strictly positive selfadjoint operator Q acting on H such that:

$$\begin{aligned} (1.3) \quad V_{23} V_{12} &= V_{12} V_{13} V_{23}, \\ V^*(Q \otimes Q)V &= Q \otimes Q, \\ (x \otimes y | V | z \otimes u) &= (\overline{z} \otimes Qy | \tilde{V} | \overline{x} \otimes Q^{-1}u) \end{aligned}$$

for any $x, z \in H$, $y \in \mathcal{D}(Q)$ and $u \in \mathcal{D}(Q^{-1})$. We say that V is a manageable multiplicative unitary V . Let

$$\begin{aligned} A &= \left\{ (\omega \otimes \text{id})V : \omega \in B(H)_* \right\}^{\text{CLS}}, \\ \hat{A} &= \left\{ (\text{id} \otimes \omega)V : \omega \in B(H)_* \right\}^{\text{CLS}} \end{aligned}$$

It is known that $A, \hat{A} \in C^*(H)$ and $V \in M(\hat{A} \otimes A)$. Moreover there exist unique coassociative morphisms $\Delta \in \text{Mor}(A, A \otimes A)$ and $\hat{\Delta} \in \text{Mor}(\hat{A}, \hat{A} \otimes \hat{A})$ such that

$$(1.4) \quad \begin{aligned} (\text{id} \otimes \Delta)V &= V_{12}V_{13}, \\ (\hat{\Delta} \otimes \text{id})V &= V_{23}V_{13}. \end{aligned}$$

We say that V is bicharacter. Elements of A should be considered as *continuous vanishing at infinity functions* on G , whereas Δ encodes the *group multiplication* on G . In short $G = (A, \Delta)$. Similarly $\hat{G} = (\hat{A}, \hat{\Delta})$. \hat{G} is the locally compact quantum group dual to G . G and \hat{G} appear in the fully symmetric manner: to pass from G to \hat{G} one has to replace V by $\hat{V} = \Sigma V^* \Sigma$, where Σ is the flip operator acting on $H \otimes H$: $\Sigma(x \otimes y) = y \otimes x$ for all $x, y \in H$.

Comparing (1.4) and (1.3) one can show that

$$(1.5) \quad \Delta(a) = V(a \otimes I)V^*$$

for any $a \in A$.

Let R be the unitary coinverse related to G . This is a linear, commuting with $*$, antimultiplicative involution

$$A \ni a \longrightarrow a^R \in A$$

such that $V^{\top \otimes R} = \tilde{V}^*$. Similarly one may introduce unitary coinverse \hat{R} related to \hat{G} . It is known that

$$V^{\hat{R} \otimes R} = V.$$

Applying antimultiplicative mapping $\hat{R} \otimes \top \otimes R$ to the both sides of (1.3) we and using the above relation we obtain

$$V^{\text{id} \otimes R \top}_{12} V^{\hat{R} \top \otimes \text{id}}_{23} = V^{\hat{R} \top \otimes \text{id}}_{23} V_{13} V^{\text{id} \otimes R \top}_{12}.$$

Extending the leg numbering notation one may rewrite the latter formula in the following form:

$$(1.6) \quad V_{1\rho} V_{\rho 3} = V_{\rho 3} V_{13} V_{1\rho},$$

where ρ and $\hat{\rho}$ are representations of A and \hat{A} acting on \overline{H} introduced by the formulae: for any $a \in A$ and $\hat{a} \in \hat{A}$

$$\begin{aligned} \rho(a) &= a^{R\top}, \\ \hat{\rho}(\hat{a}) &= \hat{a}^{\hat{R}\top}. \end{aligned}$$

Formula (1.6) means that $(\rho, \hat{\rho})$ is an anti-Heisenberg pair (cf Section 3 of [11]). Comparing (1.4) and (1.6) one can show that

$$(1.7) \quad (\text{id} \otimes \rho)\Delta(a) = \hat{V}_{1\hat{\rho}}(I \otimes \rho(a))\hat{V}_{1\hat{\rho}}^*$$

for any $a \in A$.

A linear functional ω on A is said to be normal if ω admits an extension $\tilde{\omega} \in B(H)_*$. The set of all normal functional on A will be denoted by A_* . Similarly one defines \hat{A}_* . For any $\mu, \nu \in A_*$ we set

$$\mu * \nu = (\mu \otimes \nu) \circ \Delta.$$

Using (1.5) one can easily show that $\mu * \nu \in A_*$.

An important concept of the theory of locally compact quantum groups is that of regularity and semi-regularity. It was introduced by Baaj and Skandalis in [2] and Baaj in [1]. Let

$$C = \left\{ (\text{id} \otimes \omega)(\Sigma V) : \omega \in B(H)_* \right\}^{\text{CLS}}.$$

We say that V is semi-regular if $B_0(H) \subset C$ and regular if $B_0(H) = C$. It was shown in [2] that V is regular if and only if

$$(1.8) \quad (\hat{A} \otimes I_A)V(I_{\hat{A}} \otimes A) = \hat{A} \otimes A.$$

This formula is no longer related to Hilbert space H . This is an equality of subsets of $M(\hat{A} \otimes A)$. Due to this fact regularity is the property of the group (not of a particular multiplicative unitary V used to construct the group). Applying anti-multiplicative involution $\hat{R} \otimes R$ to the both sides (1.8) we obtain equivalent condition:

$$(1.9) \quad (I_{\hat{A}} \otimes A)V(\hat{A} \otimes I_A) = \hat{A} \otimes A.$$

One can easily show that the dual of a regular group is regular. Therefore (1.9) is equivalent to

$$(1.10) \quad (I_A \otimes \hat{A})\hat{V}(A \otimes I_{\hat{A}}) = A \otimes \hat{A}.$$

2. MAIN DEFINITIONS AND RESULTS

Definition 2.1. Let D be a C^* -algebra and α be a bilinear mapping acting from $A_* \times D$ into D . We say that α is a weak right action of G on D if

$$\text{WA1. } D \text{ is generated by } \left[\alpha(\omega, D) : \omega \in A_* \right],$$

$$\text{WA2. } \alpha(\omega, \alpha(\omega', d)) = \alpha(\omega * \omega', d) \text{ for any } d \in D \text{ and } \omega, \omega' \in A_*,$$

WA3. There exist faithful representations $j \in \text{Rep}(D, K)$ and $\tilde{\alpha} \in \text{Rep}(D, K \otimes H)$ (K is a Hilbert space) such that

$$(2.1) \quad j \left(\alpha(\omega_A, d) \right) = (\text{id}_K \otimes \omega) \tilde{\alpha}(d)$$

for any $d \in D$ and $\omega \in B(H)_*$. In the above formula ω_A denotes the restriction of ω to A .

Remark: It would be nice to replace Condition WA3 by a number of algebraic and topological conditions imposed directly on the bilinear map α . This problem goes beyond the present paper. Eventually we shall return to it later.

Weakness is a weak condition for actions of quantum groups on C^* -algebras. Stronger are weak continuity and continuity [3]. We say that α is a weakly continuous action of G if it is of the form $\alpha(\omega, d) = (\text{id} \otimes \omega) \bar{\alpha}(d)$, where $\bar{\alpha} \in \text{Mor}(D, D \otimes A)$ is injective. Then for any faithful $j \in \text{Rep}(D, K)$, $\tilde{\alpha} = (j \otimes \text{id}) \bar{\alpha} \in \text{Rep}(D, K \otimes H)$ is the only representation $\tilde{\alpha}$ satisfying Condition WA3. It shows that any weakly continuous action is a weak action. If in addition the Podleś condition $\bar{\alpha}(D)(\text{I}_D \otimes A) = D \otimes A$ is satisfied then α is called continuous. Clearly continuous actions are weakly continuous.

Theorem 2.2. Let α be a weak action of a regular locally compact quantum group G on a C^* -algebra D . Then α is a continuous action.

This theorem belongs essentially to Baaĵ, Skandalis and Vaes [3]. We shall present a proof (see next section) because our setting is slightly more general than the one used in [3] and because we deal with right actions. Left actions considered by Baaĵ, Skandalis and Vaes do not require the use of the anti-Heisenberg pair.

Definition 2.3. G -dynamical system is a pair (D, α) , where D is a C^* -algebra and α is a weak right action of G on D .

Definition 2.4. G -product is a triple (B, β, ψ) , consisting of a C^* -algebra B , a left continuous action $\beta \in \text{Mor}(B, \hat{A} \otimes B)$ of \hat{G} on B and an injective morphism $\psi \in \text{Mor}(\hat{A}, B)$ such that the diagram

$$(2.2) \quad \begin{array}{ccc} \hat{A} & \xrightarrow{\psi} & B \\ \hat{\Delta} \downarrow & & \downarrow \beta \\ \hat{A} \otimes \hat{A} & \xrightarrow{\text{id} \otimes \psi} & \hat{A} \otimes B \end{array}$$

is commutative.

We recall that β is a continuous left action of \hat{G} on B is β is a faithful morphism $\beta \in \text{Mor}(B, \hat{A} \otimes B)$ such that $(\text{id} \otimes \beta)\beta = (\hat{\Delta} \otimes \text{id})\beta$ and

$$(2.3) \quad (\hat{A} \otimes \text{I}_B)\beta(B) = \hat{A} \otimes B.$$

Definition 2.5. Let (B, β, ψ) and (B', β', ψ') be G -products. We say that (B, β, ψ) and (B', β', ψ') are isomorphic if there exists an isomorphism $\iota : B \rightarrow B'$ such that the diagram

$$(2.4) \quad \begin{array}{ccccc} & & B & \xrightarrow{\beta} & \hat{A} \otimes B \\ & \nearrow \psi & \downarrow \iota & & \downarrow \text{id} \otimes \iota \\ \hat{A} & & B' & \xrightarrow{\beta'} & \hat{A} \otimes B' \\ & \searrow \psi' & & & \end{array}$$

is commutative. If moreover $M(B)$ and $M(B')$ contain the same C^* -subalgebra D and if $\iota(d) = d$ for all $d \in D$, then we say that (B, β, ψ) and (B', β', ψ') are D -isomorphic.

Definition 2.6. Let (B, β, ψ) be a G -product and $D \subset M(B)$ be a C^* -subalgebra. We say that D is a Landstad algebra for (B, β, ψ) if the following three conditions are satisfied:

C1. $\beta(d) = I_{\widehat{A}} \otimes d$ for any $d \in D$,

C2. $B = \psi(\widehat{A})D$,

C3. The C^* -algebra generated by

$$[(\text{id}_B \otimes \omega)(V_{\psi 2}(D \otimes I_A)V_{\psi 2}^*) : \omega \in A_*]$$

coincides with D .

Theorem 2.7. Any G -product admits one and only one Landstad algebra. Let (B, β, ψ) be a G -product and D be its Landstad algebra. Then the formula

$$(2.5) \quad \alpha(\omega, d) = (\text{id}_B \otimes \omega)(V_{\psi 2}(d \otimes I_A)V_{\psi 2}^*)$$

for any $\omega \in A_*$ and $d \in D$ defines a weak right action α of G on D .

Theorem 2.8 (Crossed product construction). Let (D, α) be a G -dynamical system. Then there exists G -product (B, β, ψ) and injective morphism $j \in \text{Mor}(D, B)$ such that $j(D)$ is the Landstad algebra of (B, β, ψ) and

$$(2.6) \quad j(\alpha(\omega, d)) = (\text{id}_B \otimes \omega)(V_{\psi 2}(j(d) \otimes I_A)V_{\psi 2}^*)$$

for any $\omega \in A_*$ and $d \in D$. If the action α is weakly continuous then (B, β, ψ) is unique (up to D -isomorphism).

The C^* -algebra B appearing in the above Theorem is called a crossed product of D by the action α of \widehat{G} and (if α is weakly continuous) denoted by $D \rtimes_{\alpha} \widehat{G}$. In the same case β is called the dual action and often denoted by $\widehat{\alpha}$.

3. WEAK ACTION OF A QUANTUM GROUP ON A C^* -ALGEBRA

In this section we investigate the properties of G -dynamical systems. At the end we shall prove Theorem 2.2. Let (D, α) be a G -dynamical system. Then α is a weak right action of G on D .

Let K be the Hilbert space appearing in Condition WA3. Identifying D with its copy $j(D)$ we may assume that $D \in C^*(K)$. Then formula (2.1) takes the form

$$(3.1) \quad \alpha(\omega_A, d) = (\text{id}_K \otimes \omega)\widetilde{\alpha}(d)$$

for any $d \in D$. It shows that $(\text{id}_K \otimes \omega)\widetilde{\alpha}(d)$ depends only on the restriction of ω to A . Using this observation one can easily prove that $\widetilde{\alpha}(d)$ belongs to the von Neumann tensor product $B(K) \overline{\otimes} M$, where M is the weak closure of A . Let $(\rho, \widehat{\rho})$ be the anti-Heisenberg pair acting on \overline{H} . It is known that the comultiplication Δ as well as ρ are normal mappings defined on A . Therefore they admit unique weakly continuous extensions to M and we may apply $\text{id}_K \otimes \Delta$ and $\text{id}_K \otimes \rho$ to $\widetilde{\alpha}(d)$:

$$\begin{aligned} (\text{id}_K \otimes \Delta)\widetilde{\alpha}(d) &\in B(K \otimes H \otimes H), \\ (\text{id}_K \otimes \rho)\widetilde{\alpha}(d) &\in B(K \otimes \overline{H}). \end{aligned}$$

More than that, Δ maps M into $M \overline{\otimes} M$, $(\text{id}_K \otimes \Delta)\widetilde{\alpha}(d) \in B(K) \overline{\otimes} M \overline{\otimes} M$ and we may apply $\text{id}_{K \otimes H} \otimes \rho$ to $(\text{id}_K \otimes \Delta)\widetilde{\alpha}(d)$:

$$(\text{id}_K \otimes (\text{id}_H \otimes \rho)\Delta)\widetilde{\alpha}(d) \in B(K \otimes H \otimes \overline{H}).$$

Let $\omega \in A_*$ and $d \in D$. By (3.1) and Condition WA2, for any $\omega' \in A_*$ we have

$$\begin{aligned} (\text{id}_K \otimes \omega')\widetilde{\alpha}(\alpha(\omega, d)) &= \alpha(\omega', \alpha(\omega, d)) = \alpha(\omega' * \omega, d) \\ &= (\text{id}_K \otimes \omega' * \omega)\widetilde{\alpha}(d) = (\text{id}_K \otimes \omega' \otimes \omega)(\text{id} \otimes \Delta)\widetilde{\alpha}(d). \end{aligned}$$

This formula holds for all $\omega' \in A_*$. Using (1.5) we obtain

$$\begin{aligned} (3.2) \quad \widetilde{\alpha}(\alpha(\omega, d)) &= (\text{id}_{K \otimes H} \otimes \omega)(\text{id}_K \otimes \Delta)\widetilde{\alpha}(d) \\ &= (\text{id}_{K \otimes H} \otimes \omega)(V_{23}\widetilde{\alpha}(d)_{12}V_{23}^*). \end{aligned}$$

Let $\mu \in B(\overline{H})_*$. Inserting $\omega = \mu \circ \rho$, and using (1.7) we get

$$(3.3) \quad \begin{aligned} \tilde{\alpha}(\alpha(\mu \circ \rho, d)) &= (\text{id}_{K \otimes H} \otimes \mu)(\text{id}_K \otimes (\text{id}_H \otimes \rho)\Delta)\tilde{\alpha}(d) \\ &= (\text{id}_{K \otimes H} \otimes \mu) \left(\widehat{V}_{1\widehat{\rho}}\tilde{\alpha}(d)_{0\rho}\widehat{V}_{1\widehat{\rho}}^* \right). \end{aligned}$$

Let

$$(3.4) \quad D_1 = \left[\alpha(\omega, D) : \omega \in B(H)_* \right].$$

Clearly D_1 is invariant under hermitian conjugation. By Condition WA1, C^* -algebra D is generated by D_1 :

$$D = \left[D_k : k = 1, 2, 3, \dots \right],$$

where $D_2 = D_1 D_1$, $D_3 = D_1 D_1 D_1$ and so on. We shall use formula (3.3). Taking the closed linear span over all $\mu \in B(\overline{H})_*$ and $d \in D$ we obtain

$$(3.5) \quad \begin{aligned} \tilde{\alpha}(D_1) &= \left[(\text{id}_{K \otimes H} \otimes \mu) ((\text{id}_K \otimes (\text{id}_A \otimes \rho)\Delta)\tilde{\alpha}(D)) : \mu \in B(\overline{H})_* \right] \\ &= \left[(\text{id}_{K \otimes H} \otimes \mu) \left(\widehat{V}_{1\widehat{\rho}}\tilde{\alpha}(D)_{0\rho}\widehat{V}_{1\widehat{\rho}}^* \right) : \mu \in B(\overline{H})_* \right]. \end{aligned}$$

Now we are able to present the prove of Baaj-Skandalis-Vaes theorem.

Proof of Theorem 2.2. We compute $\tilde{\alpha}(D_1)(I_K \otimes A)$. We know that unitary \widehat{V}^* belongs to $M(A \otimes \widehat{A})$. Therefore $\widehat{V}^*(A \otimes \widehat{A}) = A \otimes \widehat{A}$. In the following computation we use (1.2): at first (in the second equality) μ emits $\widehat{\rho}(\widehat{A})$ to the left, later (in the forth equality) μ absorbs $\widehat{\rho}(\widehat{A})$ back. Finally in the last equality μ emits $\widehat{\rho}(\widehat{A})$ to the right.

$$\begin{aligned} \tilde{\alpha}(D_1)(I_K \otimes A) &= \left[(\text{id}_{K \otimes H} \otimes \mu) \left(\widehat{V}_{1\widehat{\rho}}\tilde{\alpha}(D)_{0\rho}\widehat{V}_{1\widehat{\rho}}^*(I_K \otimes A \otimes I_{\overline{H}}) \right) : \mu \in B(\overline{H})_* \right] \\ &= \left[(\text{id}_{K \otimes H} \otimes \mu) \left(\widehat{V}_{1\widehat{\rho}}\tilde{\alpha}(D)_{0\rho} \left(\widehat{V}^*(A \otimes \widehat{A}) \right)_{1\widehat{\rho}} \right) : \mu \in B(\overline{H})_* \right] \\ &= \left[(\text{id}_{K \otimes H} \otimes \mu) \left(\widehat{V}_{1\widehat{\rho}}\tilde{\alpha}(D)_{0\rho} (A \otimes \widehat{A})_{1\widehat{\rho}} \right) : \mu \in B(\overline{H})_* \right] \\ &= \left[(\text{id}_{K \otimes H} \otimes \mu) \left(\left(\widehat{V}(A \otimes I_{\widehat{A}}) \right)_{1\widehat{\rho}} \tilde{\alpha}(D)_{0\rho} \right) : \mu \in B(\overline{H})_* \right] \\ &= \left[(\text{id}_{K \otimes H} \otimes \mu) \left(\left((I_A \otimes \widehat{A})\widehat{V}(A \otimes I_{\widehat{A}}) \right)_{1\widehat{\rho}} \tilde{\alpha}(D)_{0\rho} \right) : \mu \in B(\overline{H})_* \right]. \end{aligned}$$

In this theorem we assume that G is regular. Therefore (cf (1.10)) $(I_A \otimes \widehat{A})\widehat{V}(A \otimes I_{\widehat{A}}) = A \otimes \widehat{A}$ and

$$\begin{aligned} \tilde{\alpha}(D_1)(I_K \otimes A) &= \left[(\text{id}_{K \otimes H} \otimes \mu) \left((A \otimes \widehat{A})_{1\widehat{\rho}} \tilde{\alpha}(D)_{0\rho} \right) : \mu \in B(\overline{H})_* \right] \\ &= \left[(\text{id}_{K \otimes H} \otimes \mu) \left((A \otimes I_{\overline{H}})_{12} \tilde{\alpha}(D)_{0\rho} \right) : \mu \in B(\overline{H})_* \right] \\ &= \left[(\text{id}_{K \otimes H} \otimes \omega) \left((I_K \otimes A \otimes I_H)\tilde{\alpha}(D)_{02} \right) : \omega \in A_* \right] \\ &= \left[\alpha(\omega, D) : \omega \in A_* \right] \otimes A = D_1 \otimes A. \end{aligned}$$

In the second equality μ absorbs $\widehat{\rho}(\widehat{A})$, in the third equality $\mu \circ \rho$ is replaced by ω , next we use (3.1) and (3.4).

We showed that

$$\tilde{\alpha}(D_1)(I \otimes A) = D_1 \otimes A.$$

Now we have:

$$\tilde{\alpha}(D_2)(I \otimes A) = \tilde{\alpha}(D_1)\tilde{\alpha}(D_1)(I \otimes A) = \tilde{\alpha}(D_1)(I \otimes A)(D_1 \otimes I) = D_2 \otimes A.$$

In the same way one can show that

$$\tilde{\alpha}(D_k)(I \otimes A) = D_k \otimes A.$$

for $k = 3, 4, 5, \dots$. Taking the closed linear span over all natural k we obtain Podleś condition:

$$\tilde{\alpha}(D)(I \otimes A) = D \otimes A.$$

Clearly it implies that $\tilde{\alpha} \in \text{Mor}(D, D \otimes A)$. This way we showed that action α is continuous. \square

4. EXISTENCE OF CROSSED PRODUCT ALGEBRA

This section is devoted to the proof of the main part of Theorem 2.8. We shall keep notation introduced in the previous section. It turns out that the crossed product of D by the action α of G may be identified with $B = \widehat{\Delta}(\widehat{A})_{13}(I_{\widehat{A}} \otimes \tilde{\alpha}(D))$. However it is not clear that this formula defines a C^* -algebra. To prove this fact we shall use the following technical result:

Proposition 4.1. *Let $X, Y \in C^*(L)$, $\tilde{X}, \tilde{Y} \in C^*(L \otimes \overline{H})$ (L and \overline{H} are Hilbert spaces) and*

$$(4.1) \quad \left\{ \begin{array}{l} X = \left\{ \begin{array}{l} \text{The } C^* \text{-algebra generated by} \\ [(\text{id}_L \otimes \mu)\tilde{X} : \mu \in B(\overline{H})_*] \end{array} \right\}, \\ \tilde{Y}(I_L \otimes B_0(\overline{H})) = Y \otimes B_0(\overline{H}), \end{array} \right.$$

Assume that $\tilde{X}\tilde{Y}$ is a C^ -algebra. Then*

$$(4.2) \quad XY = \left\{ \begin{array}{l} \text{The } C^* \text{-algebra generated by} \\ [(\text{id}_L \otimes \mu)\tilde{X}\tilde{Y} : \mu \in B(\overline{H})_*] \end{array} \right\}.$$

Proof. Let

$$\begin{aligned} X_1 &= [(\text{id}_L \otimes \mu)\tilde{X} : \mu \in B(\overline{H})_*] \\ Z_1 &= [(\text{id}_L \otimes \mu)\tilde{X}\tilde{Y} : \mu \in B(\overline{H})_*] \end{aligned}$$

Then X is generated by X_1 . Clearly X_1 and Z_1 are invariant under hermitian conjugation. In the computation below, first μ emits $B_0(\overline{H})$, next we use second relation of (4.1) and finally μ absorbs $B_0(\overline{H})$:

$$\begin{aligned} Z_1 &= [(\text{id}_L \otimes \mu)(\tilde{X}\tilde{Y}(I_L \otimes B_0(\overline{H}))) : \mu \in B(\overline{H})_*] \\ &= [(\text{id}_L \otimes \mu)(\tilde{X}(Y \otimes B_0(\overline{H}))) : \mu \in B(\overline{H})_*] \\ &= [(\text{id}_L \otimes \mu)(\tilde{X}(I_L \otimes B_0(\overline{H}))) : \mu \in B(\overline{H})_*] \quad Y = X_1 Y \end{aligned}$$

Applying to the both sides hermitian conjugation we get:

$$Z_1 = X_1 Y = Y X_1.$$

Now we have: $Z_1 Z_1 = X_1 Y X_1 Y = X_1 X_1 Y Y = X_1 X_1 Y$. In the same way $Z_k = X_k Y$, where Z_k (X_k respectively) is the product of k copies of Z_1 (X_1 respectively), $k = 3, 4, \dots$. Taking the closed linear span over all natural k we obtain (4.2). \square

Let $(\rho, \hat{\rho})$ be the anti-Heisenberg pair acting on a Hilbert space \overline{H} . Then the commutator

$$(4.3) \quad [(\text{id}_A \otimes \rho)\Delta(a), (\text{id}_{\widehat{A}} \otimes \hat{\rho})\widehat{\Delta}(\hat{a})] = 0$$

for any $a \in A$ and $\hat{a} \in \widehat{A}$. This formula is a special case of formula (3.7) of [11].

We shall use Proposition 4.1 in the following context. Let $L = H \otimes K \otimes H$ and

$$\tilde{X} = I_H \otimes (\text{id}_K \otimes (\text{id}_H \otimes \rho)\Delta) \tilde{\alpha}(D), \quad X = I_H \otimes \tilde{\alpha}(D),$$

$$\tilde{Y} = (\widehat{\Delta}_{02} \otimes \hat{\rho}) \widehat{\Delta}(\widehat{A}), \quad Y = \widehat{\Delta}(\widehat{A})_{02}.$$

Then $\tilde{X}, \tilde{Y} \in C^*(H \otimes K \otimes H \otimes \overline{H}) = C^*(L \otimes \overline{H})$ and $X, Y \in C^*(H \otimes K \otimes H) = C^*(L)$. We shall prove that these algebras satisfy the two relations (4.1).

First formula of (4.1) follows immediately from (3.5) and Condition WA1. To prove the second formula we use the cancelation property $\widehat{\Delta}(\widehat{A})(I_H \otimes \widehat{A}) = \widehat{A} \otimes \widehat{A}$. Applying to the both sides $\widehat{\Delta}_{02} \otimes \hat{\rho}$ and multiplying from the right by $I_{H \otimes K \otimes H} \otimes B_0(\overline{H})$ we get the second relation of (4.1).

By coassociativity of $\widehat{\Delta}$ we have

$$\widetilde{Y} = \left(\text{id}_H \otimes (\text{id}_H \otimes \widehat{\rho}) \widehat{\Delta} \right) \widehat{\Delta}(\widehat{A})_{134}.$$

Formula (4.3) shows now that elements of \widetilde{X} commute with elements of \widetilde{Y} . Therefore $\widetilde{X}\widetilde{Y} = \widetilde{Y}\widetilde{X}$ and $\widetilde{X}\widetilde{Y}$ is a C^* -algebra. Proposition 4.1 shows now that

$$(4.4) \quad B = \widehat{\Delta}(\widehat{A})_{02}(\text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(D)).$$

is a C^* -algebra. Clearly $B \in C^*(H \otimes K \otimes H)$.

For any $\widehat{a} \in \widehat{A}$ we set

$$\psi(\widehat{a}) = \widehat{\Delta}(\widehat{a})_{02}.$$

Then ψ is a faithful representation of \widehat{A} acting on $H \otimes K \otimes H$. With this notation

$$(4.5) \quad B = \psi(\widehat{A})(\text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(D)).$$

The reader should notice that $B \subset M(\widehat{A} \otimes B_0(K \otimes H))$. Therefore the formula

$$\beta(b) = (\widehat{\Delta} \otimes \text{id}_H \otimes \text{id}_K)(b),$$

where $b \in B$, defines a faithful representation of B acting on $H \otimes H \otimes K \otimes H$. Let us notice that

$$(4.6) \quad \begin{aligned} \beta(\psi(\widehat{a})) &= (\text{id}_H \otimes \psi) \widehat{\Delta}(\widehat{a}) \\ \beta(\text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(d)) &= \text{I}_{\widehat{A}} \otimes \text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(d) \end{aligned}$$

for any $\widehat{a} \in \widehat{A}$ and $d \in D$. The second formula is obvious, the first one follows from coassociativity of $\widehat{\Delta}$.

We have $\psi(\widehat{A})B = \psi(\widehat{A})\psi(\widehat{A})(\text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(D)) = \psi(\widehat{A})(\text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(D)) = B$. It shows that $\psi \in \text{Mor}(A, B)$. Similarly $\beta(B) = (\text{id}_H \otimes \psi) \widehat{\Delta}(\widehat{A})(\text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(D))$ and

$$\begin{aligned} (\widehat{A} \otimes \text{I}_B)\beta(B) &= (\text{id}_H \otimes \psi) \left((\widehat{A} \otimes \text{I}_{\widehat{A}}) \widehat{\Delta}(\widehat{A}) \right) (\text{I}_{\widehat{A}} \otimes \text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(D)) \\ &= (\text{id} \otimes \psi)(\widehat{A} \otimes \widehat{A})(\text{I}_{\widehat{A}} \otimes \text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(D)) = \widehat{A} \otimes B. \end{aligned}$$

So we have

$$(4.7) \quad (\widehat{A} \otimes \text{I}_B)\beta(B) = \widehat{A} \otimes B.$$

Multiplying (from the left) by $\text{I}_{\widehat{A}} \otimes B$ we obtain $(A \otimes B)\beta(B) = A \otimes B$. It shows that $\beta \in \text{Mor}(B, A \otimes B)$.

Using the coassociativity of $\widehat{\Delta}$ one can check that $(\widehat{\Delta} \otimes \text{id}_B)\beta(b) = (\text{id}_{\widehat{A}} \otimes \beta)\beta(b)$. It means that β is a continuous left action of G on B . (4.7) is the Podleś condition for this action. First formula of (4.6) shows that the diagram (2.2) is commutative. Hence (B, β, ψ) is a G -product.

For any $d \in D$ we set

$$j(d) = \text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(d).$$

Then $j \in \text{Mor}(D, B)$. Let RHS be the right hand side of (2.6). Then

$$\begin{aligned} \text{RHS} &= (\text{id}_B \otimes \omega)(V_{\psi 4} \widetilde{\alpha}(d)_{23} V_{\psi 4}^*) \\ &= (\text{id}_{\widehat{A}} \otimes \text{id}_K \otimes \text{id}_H \otimes \omega)(V_{34} V_{14} \widetilde{\alpha}(d)_{23} V_{14}^* V_{34}^*) \\ &= \text{I}_{\widehat{A}} \otimes (\text{id}_K \otimes \text{id}_H \otimes \omega)(V_{23} \widetilde{\alpha}(d)_{12} V_{23}^*) \\ &= \text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(\alpha(\omega, d)) = j(\alpha(\omega, d)), \end{aligned}$$

where in the last step we used (3.2). Formula (2.6) is verified. To end the proof of the first part of Theorem 2.8 we have to show that $j(D) = \text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(D)$ is a Landstad algebra of (B, β, ψ) .

In the present context Landstad-Vaes conditions take the form:

$$\text{C1'}. \quad \beta(\text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(d)) = \text{I}_{\widehat{A}} \otimes \text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(d) \text{ for any } d \in D,$$

$$\text{C2'}. \quad B = \psi(\widehat{A})(\text{I}_{\widehat{A}} \otimes \widetilde{\alpha}(D)),$$

$$\text{C3'}. \quad \text{The } C^* \text{-algebra generated by } [j(\alpha(\omega, D)) : \omega \in A_*] \text{ coincides with } j(D).$$

Conditions C1' and C2' are already verified (cf (4.5) and the second formula of (4.6)). Condition C3' follows immediately from Condition WA1 of Definition 2.1. The proof of the first part of Theorem 2.8 is complete. The proof of the remaining part is given at the end of section 5.

5. EXISTENCE AND UNIQUENESS OF LANDSTAD ALGEBRA

In this section we consider a G -product (B, β, ψ) . We shall follow the path elaborated by S. Vaes in [18, Proof of Thm 6.7].

Proposition 5.1. *Let (B, β, ψ) be a G -product, $\varphi \in \text{Mor}(B, B_0(H) \otimes B)$ be an injective morphism introduced by the formula: for any $b \in B$*

$$(5.1) \quad \varphi(b) = \widehat{V}_{1\psi}^* \beta(b) \widehat{V}_{1\psi}.$$

and let

$$(5.2) \quad D_1 = \left[(\omega \otimes \text{id}_B) \varphi(B) : \omega \in B(H)_* \right],$$

Then

1. D_1 be a norm closed linear subset of $M(B)$. Elements of D_1 are β -invariant:

$$\beta(d) = I_{\widehat{A}} \otimes d$$

for any $d \in D_1$,

2. $B = \psi(\widehat{A})D_1$.

Proof. At first we shall show that

$$(5.3) \quad \begin{aligned} \varphi \circ \psi(\widehat{a}) &= \widehat{a} \otimes I_B, \\ (\text{id}_H \otimes \beta) \varphi(b) &= \varphi(b)_{13} \end{aligned}$$

for any $\widehat{a} \in \widehat{A}$ and $b \in B$.

Indeed, for any $\widehat{a} \in \widehat{A}$ we have:

$$\begin{aligned} \varphi \circ \psi(\widehat{a}) &= \widehat{V}_{1\psi}^* \left[\beta \circ \psi(\widehat{a}) \right] \widehat{V}_{1\psi} = \widehat{V}_{1\psi}^* \left[(\text{id}_{\widehat{A}} \otimes \psi) \widehat{\Delta}(\widehat{a}) \right] \widehat{V}_{1\psi} \\ &= (\text{id}_{\widehat{A}} \otimes \psi) \left(\widehat{V}^* \widehat{\Delta}(\widehat{a}) \widehat{V} \right) = (\text{id}_{\widehat{A}} \otimes \psi) (\widehat{a} \otimes I_{\widehat{A}}) = \widehat{a} \otimes I_B \end{aligned}$$

In the above computation we used the commutativity of (2.2) and formula (1.5) (with G replaced by \widehat{G}). Now let $b \in B$. To compute

$$(\text{id}_H \otimes \beta) \varphi(b) = (\text{id}_H \otimes \beta) \widehat{V}_{1\psi}^* (\text{id}_H \otimes \beta) \beta(b) (\text{id}_H \otimes \beta) \widehat{V}_{1\psi}$$

we notice that

$$\begin{aligned} (\text{id}_H \otimes \beta) \widehat{V}_{1\psi} &= (\text{id}_H \otimes \beta \circ \psi) \widehat{V} = \left(\text{id}_H \otimes (\text{id}_{\widehat{A}} \otimes \psi) \widehat{\Delta} \right) \widehat{V} = \widehat{V}_{12} \widehat{V}_{1\psi}, \\ (\text{id}_H \otimes \beta) \beta(b) &= (\widehat{\Delta} \otimes \text{id}_B) \beta(b) = \widehat{V}_{12} \beta(b)_{13} \widehat{V}_{12}^*. \end{aligned}$$

Therefore

$$\begin{aligned} (\text{id}_H \otimes \beta) \varphi(b) &= \widehat{V}_{1\psi}^* \widehat{V}_{12}^* \widehat{V}_{12} \beta(b)_{13} \widehat{V}_{12}^* \widehat{V}_{12} \widehat{V}_{1\psi} \\ &= \widehat{V}_{1\psi}^* \beta(b)_{13} \widehat{V}_{1\psi} = \varphi(b)_{13} \end{aligned}$$

and formulae (5.3) are proven. Second formula of (5.3) immediately implies that slices of $\varphi(b)$ are β -invariant and Statement 1 follows. We shall prove Statement 2.

We know that \widehat{V} is a unitary multiplier of $B_0(H) \otimes \widehat{A}$. Therefore $\widehat{V}_{1\psi}$ is a unitary multiplier of $B_0(H) \otimes B$ and

$$\begin{aligned} (B_0(H) \otimes \widehat{A}) \widehat{V}^* &= B_0(H) \otimes \widehat{A} \\ (B_0(H) \otimes B) \widehat{V}_{1\psi} &= B_0(H) \otimes B \end{aligned}$$

Moreover Podleś condition (2.3) easily shows that

$$(B_0(H) \otimes I_B) \beta(B) = B_0(H) \otimes B.$$

Using the above formulae we obtain

$$\begin{aligned}
 (B_0(H) \otimes \psi(\hat{A}))\varphi(B) &= (B_0(H) \otimes \psi(\hat{A}))\hat{V}_{1\psi}^* \beta(B) \hat{V}_{1\psi} \\
 &= (\text{id}_H \otimes \psi) \left((B_0(H) \otimes \hat{A}) \hat{V}^* \right) \beta(B) \hat{V}_{1\psi} \\
 &= (\text{id}_H \otimes \psi) \left(B_0(H) \otimes \hat{A} \right) \beta(B) \hat{V}_{1\psi} \\
 &= (I_H \otimes \psi(\hat{A})) \left(B_0(H) \otimes I_B \right) \beta(B) \hat{V}_{1\psi} \\
 &= (I_H \otimes \psi(\hat{A})) \left(B_0(H) \otimes B \right) \hat{V}_{1\psi} \\
 &= B_0(H) \otimes \psi(\hat{A})B.
 \end{aligned}$$

Taking into account the equality $\psi(\hat{A})B = B$ (this is because $\psi \in \text{Mor}(\hat{A}, B)$) we get

$$(B_0(H) \otimes \psi(\hat{A}))\varphi(B) = B_0(H) \otimes B.$$

Now, using (1.2) we obtain

$$\begin{aligned}
 \psi(\hat{A})D_1 &= \left[\psi(\hat{A})(\omega \otimes \text{id}_B)\varphi(B) : \omega \in B(H)_* \right] \\
 &= \left[(\omega \otimes \text{id}_B) \left\{ (B_0(H) \otimes \psi(\hat{A}))\varphi(B) \right\} : \omega \in B(H)_* \right] \\
 &= \left[(\omega \otimes \text{id}_B) \left\{ B_0(H) \otimes B \right\} : \omega \in B(H)_* \right] = B
 \end{aligned}$$

Statement 2 is proven. \square

With the notation introduced in the previous Theorem we have

Proposition 5.2. *C^* -algebra generated by D_1 is the only Landstad algebra of (B, β, ψ) .*

Proof. Let D be a C^* -subalgebra of B satisfying conditions C1 and C2. Condition C1 shows that $\beta(D) = I_{\hat{A}} \otimes D$. Therefore $\varphi(D) = \hat{V}_{1\psi}^* (I_{\hat{A}} \otimes D) \hat{V}_{1\psi}$. We know that $\omega \in A_*$ if and only if ω is restriction (to A) of a normal state on $B(H)$. Therefore

$$\begin{aligned}
 \left[(\omega \otimes \text{id}_B) \left(\hat{V}_{1\psi}^* (I_{\hat{A}} \otimes D) \hat{V}_{1\psi} \right) : \omega \in A_* \right] &= [(\omega \otimes \text{id}_B)\varphi(D) : \omega \in A_*] \\
 &= [(\omega \otimes \text{id}_B)\varphi(D) : \omega \in B(H)_*]
 \end{aligned}$$

We shall prove that the latter set coincides with D_1 . Indeed:

$$\begin{aligned}
 [(\omega \otimes \text{id}_B)\varphi(D) : \omega \in B(H)_*] &= [(\omega \otimes \text{id}_B) \left((\hat{A} \otimes I_B)\varphi(D) \right) : \omega \in B(H)_*] \\
 &= [(\omega \otimes \text{id}_B)\varphi(\psi(\hat{A})D) : \omega \in B(H)_*] \\
 &= [(\omega \otimes \text{id}_B)\varphi(B) : \omega \in B(H)_*] = D_1.
 \end{aligned}$$

The first equality follows from (1.2), the second one from first relation of (5.3) and the third one from C2. We showed that

$$(5.4) \quad \left[(\omega \otimes \text{id}_B) \left(\hat{V}_{1\psi}^* (I_{\hat{A}} \otimes D) \hat{V}_{1\psi} \right) : \omega \in A_* \right] = D_1.$$

Assume now that D is a Landstad algebra for (B, β, ψ) . Then Condition C3 shows that D is generated by D_1 .

Conversely assume that D is the C^* -algebra generated by D_1 . Then $D \subset M(B)$. A moment of reflection shows that

$$D = \left(D_1 \cup D_1 D \right)^{\text{CLS}}$$

We know that $\psi(\hat{A})D_1 = B$ (see Statement 2 of Proposition 5.1). The simple observation: $\psi(\hat{A})D_1 D = B D \subset B$ combined with the above formula shows that $\psi(\hat{A})D = B$. Condition C2 is verified.

According to Statement 1 of Proposition 5.1 elements of D_1 are β -invariant. The same is true for elements of D and Condition C1 is verified.

Now we may use (5.4). Remembering that D is generated by D_1 we see that Condition C3 is satisfied. It means that D is a Landstad algebra for (B, β, ψ) .

The existence and uniqueness of Landstad algebra is shown. \square

Now we are able to prove Theorem 2.7

Proof. It follows immediately from C3, that formula (2.5) defines a mapping α acting from $A_* \times D$ into D satisfying Condition WA1 of Definition 2.1. We shall check Condition WA3.

One may assume that $B \in C^*(K)$, where K is a Hilbert space. Then $D \subset M(B) \subset B(K)$. Denote by j the embedding of D into $B(K)$ and by $\tilde{\alpha}$ the representation of D acting on $K \otimes H$ introduced by the formula

$$\tilde{\alpha}(d) = V_{\psi 2}(d \otimes I_A)V_{\psi 2}^*$$

for any $d \in D$. Then (2.5) coincides with (2.1) and Condition WA3 is verified.

Clearly $V_{\psi 2} \in M(B_0(K) \otimes A)$ and $\tilde{\alpha}(d) \in M(B_0(K) \otimes A)$. Using the bicharacter formula (1.4) we obtain

$$(\text{id}_K \otimes \Delta)\tilde{\alpha}(d) = V_{\psi 2}V_{\psi 3}(d \otimes I_A \otimes I_A)V_{\psi 3}^*V_{\psi 2}^*.$$

Therefore for any $\omega \in A_*$ we have

$$(\text{id}_{K \otimes H} \otimes \omega)(\text{id}_K \otimes \Delta)\tilde{\alpha}(d) = V_{\psi 2}(\alpha(\omega, d) \otimes I_A)V_{\psi 2}^* = \tilde{\alpha}(\alpha(\omega, d)).$$

Now we are able to verify Condition WA2. Let $\omega' \in \hat{A}_*$. Then

$$\begin{aligned} \alpha(\omega', \alpha(\omega, d)) &= (\text{id}_K \otimes \omega')\tilde{\alpha}(\alpha(\omega, d)) = (\text{id}_K \otimes \omega' \otimes \omega)(\text{id}_K \otimes \Delta)\tilde{\alpha}(d) \\ &= (\text{id}_K \otimes \omega' * \omega)\tilde{\alpha}(d) = \alpha(\omega' * \omega, d). \end{aligned}$$

We have shown that α is a weak right action of G on D . The proof of Theorem 2.7 is complete. \square

We end this section with the proof of the last statement of Theorem 2.8. Assume that the action α is weakly continuous of G on a C^* -algebra D . Then there exists $\bar{\alpha} \in \text{Mor}(D, D \otimes A)$ such that

$$(5.5) \quad \alpha(\omega, d) = (\text{id}_K \otimes \omega)\bar{\alpha}(d),$$

for any $\omega \in A_*$ and $d \in D$.

Let (B, β, ψ) be a G -product and $j \in \text{Mor}(D, B)$ be an injective morphism such that $j(D)$ is the Landstad algebra of (B, β, ψ) . Assume that formula (2.6) holds.

Let $d \in D$. By Condition C1, $\beta(j(d)) = I_{\hat{A}}j(d)$. Comparing (2.6) with (5.5) and using (5.1) we obtain

$$(j \otimes \text{id}_H)\bar{\alpha}(d) = V_{\psi 2}(j(d) \otimes I_A)V_{\psi 2}^* = \text{flip} \left(\hat{V}_{1\psi}^* \beta(j(d)) \hat{V}_{1\psi} \right) = \text{flip}(\varphi(j(d))).$$

Let $\hat{a} \in \hat{A}$. First formula of (5.3) shows that

$$(j \otimes \text{id}_H)(I_D \otimes \hat{a}) = \text{flip}(\varphi(\psi(\hat{a}))).$$

We know (see Condition C2) that $\psi(\hat{A})j(D) = B$. Therefore

$$(j \otimes \text{id}_H) \left((I_D \otimes \hat{A})\bar{\alpha}(D) \right) = \text{flip} \circ \varphi(B).$$

Remembering that morphisms j , flip and φ are injective we conclude that $(I_D \otimes \hat{A})\bar{\alpha}(D)$ is a C^* -algebra and that B is isomorphic to $(I_D \otimes \hat{A})\bar{\alpha}(D)$. This isomorphism maps $j(d)$ onto $\bar{\alpha}(d)$ and $\psi(\hat{a})$ onto $I_D \otimes \hat{a}$.

If (B', β', ψ') is another G -product and $j' \in \text{Mor}(D, B')$ is an injective morphism such that $j'(D)$ is the Landstad algebra of (B', β', ψ') , then B' is isomorphic to $(I_D \otimes \hat{A})\bar{\alpha}(D)$ and the isomorphism maps $j'(d)$ onto $\bar{\alpha}(d)$ and $\psi'(\hat{a})$ onto $I_D \otimes \hat{a}$. Combining the two isomorphisms we obtain an isomorphism

$$\iota : B \longrightarrow B'$$

such that $\iota(j(d)) = j'(d)$ and $\iota(\psi(\hat{a})) = \psi'(\hat{a})$ for any $\hat{a} \in \hat{A}$ and $d \in D$. Now, using (2.2) and Conditions C1 and C2 one can easily show that $(\text{id}_{\hat{A}} \otimes \iota) \circ \beta = \beta' \circ \iota$. It shows that (B', β', ψ') and (B, β, ψ) are D -isomorphic. The proof of Theorem 2.8 is complete.

6. EXAMPLES OF G -PRODUCTS

The following example comes from the Kasprzak theory [5, 6, 7]. He was able to find an elegant realisation of Rieffel deformation [14] of C^* -algebras endowed with an action of a group. Kasprzak (and Rieffel) worked with locally compact abelian group, but due to the further developement we may consider any locally compact quantum group G . We shall use the notation introduced in previous sections.

Assume that we have a unitary two-cocycle. This is a unitary element $\Omega \in M(\widehat{A} \otimes \widehat{A})$ such that

$$(6.1) \quad (\Omega \otimes I)(\widehat{\Delta} \otimes \text{id})(\Omega) = (I \otimes \Omega)(\text{id} \otimes \widehat{\Delta})(\Omega).$$

We shall also assume that the Drinfeld twist induced by Ω is trivial:

$$(6.2) \quad \Omega^* \widehat{\Delta}(\widehat{a}) \Omega = \widehat{\Delta}(\widehat{a})$$

for any $\widehat{a} \in \widehat{A}$.

Let D be a C^* -algebra equipped with a right weak action of G . Then (D, α) be a G -dynamical system. Using Theorem 2.8 we may find G -product (B, β, ψ) with Landstad algebra isomorphic to D .

Theorem 6.1. *For any $b \in B$ we set*

$$\beta'(b) = \Omega_{1\psi}^* \beta(b) \Omega_{1\psi}.$$

Then $\beta'(b) \in M(\widehat{A} \otimes B)$, $\beta' \in \text{Mor}(B, \widehat{A} \otimes B)$ is a continuous left action of \widehat{G} on B and (B, β', ψ) is a G -product.

Proof. We have to show that

$$(\star) \quad (\text{id} \otimes \beta')\beta'(b) = (\widehat{\Delta} \otimes \text{id})\beta'(b),$$

$$(\star\star) \quad (\widehat{A} \otimes I_B)\beta'(B) = \widehat{A} \otimes B,$$

$$(\star\star\star) \quad \beta'(\psi(\widehat{a})) = (\text{id} \otimes \psi)\widehat{\Delta}(\widehat{a}).$$

Ad. $(\star\star\star)$. We have:

$$\begin{aligned} \beta'(\psi(\widehat{a})) &= \Omega_{1\psi}^* \beta(\psi(\widehat{a})) \Omega_{1\psi} = \Omega_{1\psi}^* (\text{id} \otimes \psi)\widehat{\Delta}(\widehat{a}) \Omega_{1\psi} \\ &= (\text{id} \otimes \psi) \left(\Omega^* \widehat{\Delta}(\widehat{a}) \Omega \right) = (\text{id} \otimes \psi)\widehat{\Delta}(\widehat{a}). \end{aligned}$$

Ad. (\star) . We already know that $\Omega_{1, \beta'\psi} = (\text{id} \otimes \beta'\psi)\Omega = (\text{id} \otimes (\text{id} \otimes \psi)\widehat{\Delta})\Omega = \Omega_{1, (\text{id} \otimes \psi)\widehat{\Delta}}$. We compute

$$(\text{id} \otimes \beta')\beta'(b) = \Omega_{1, \beta'\psi}^* (\text{id} \otimes \beta')\beta(b) \Omega_{1, \beta'\psi} = \Omega_{1, \beta'\psi}^* \Omega_{2\psi}^* (\text{id} \otimes \beta)\beta(b) \Omega_{2\psi} \Omega_{1, \beta'\psi}.$$

On the other hand

$$(\widehat{\Delta} \otimes \text{id})\beta'(b) = \Omega_{\widehat{\Delta}, \psi}^* (\widehat{\Delta} \otimes \text{id})\beta(b) \Omega_{\widehat{\Delta}, \psi}.$$

To prove the statement it is enough to show that $\Omega_{2\psi} \Omega_{1, \beta'\psi} \Omega_{\widehat{\Delta}, \psi}^*$ commutes with $(\widehat{\Delta} \otimes \text{id})\beta(b)$. Using at the last moment (6.1) we obtain

$$\Omega_{2\psi} \Omega_{1, \beta'\psi} \Omega_{\widehat{\Delta}, \psi}^* = \Omega_{2\psi} \Omega_{1, (\text{id} \otimes \psi)\widehat{\Delta}} \Omega_{\widehat{\Delta}, \psi}^* = (\text{id} \otimes \text{id} \otimes \psi) \left((I \otimes \Omega)(\text{id} \otimes \widehat{\Delta})\Omega(\widehat{\Delta} \otimes \text{id})\Omega^* \right) = \Omega \otimes I.$$

Now the commutativity follows immediately from (6.2).

Ad. $(\star\star)$. Ω is a unitary multiplier of $\widehat{A} \otimes \widehat{A}$. Therefore $(\widehat{A} \otimes \widehat{A})\Omega^* = \widehat{A} \otimes \widehat{A}$. Using the cancelation formula $\widehat{A} \otimes \widehat{A} = (\widehat{A} \otimes I)\widehat{\Delta}(\widehat{A})$ and relation (6.2) we obtain

$$(\widehat{A} \otimes I)\Omega^* \widehat{\Delta}(\widehat{A}) = (\widehat{A} \otimes I)\widehat{\Delta}(\widehat{A}).$$

Applying $\text{id} \otimes \psi$ to the both sides and taking into account commutativity of (2.2) we get

$$(\widehat{A} \otimes I)\Omega_{1\psi}^* \beta(\psi(\widehat{A})) = (\widehat{A} \otimes I)\beta(\psi(\widehat{A})).$$

We know that $\psi \in \text{Mor}(\widehat{A}, B)$. Therefore $\psi(\widehat{A})B = B$. Multiplying both sides of the above formula by $\beta(B)$ and using Podleś condition for the action β we have

$$(\widehat{A} \otimes I)\Omega_{1\psi}^* \beta(B) = (\widehat{A} \otimes I)\beta(B) = \widehat{A} \otimes B.$$

Finally

$$(\widehat{A} \otimes I)\beta'(B) = (\widehat{A} \otimes B)\Omega_{1\psi} = \widehat{A} \otimes B.$$

□

Let D' be the Landstad algebra related to G -product (B, β', ψ) . According to Kasprzak, D' may be considered as Rieffel deformation of D . Recently Kasprzak theory was extended by Neshveyev and Tuset [12] for non-trivial Drinfeld twist i.e. when (6.2) does not hold. Then they had to consider deformations of G and ψ .

Interesting examples of G -products are provided by the following Theorem:

Theorem 6.2. *Let X be a C^* -algebra and U be a unitary element of $M(X \otimes A)$ such that*

$$(6.3) \quad (\text{id}_X \otimes \Delta)U = U_{12}U_{13}.$$

For any $\hat{a} \in \hat{A}$ we set

$$(6.4) \quad \psi(\hat{a}) = \hat{U}(\hat{a} \otimes \text{I}_X)\hat{U}^*,$$

where $\hat{U} = \text{flip}(U^)$ is a unitary element of $M(A \otimes X)$. Then $\psi \in \text{Mor}(\hat{A}, \hat{A} \otimes X)$ and the diagram*

$$(6.5) \quad \begin{array}{ccc} \hat{A} & \xrightarrow{\psi} & \hat{A} \otimes X \\ \hat{\Delta} \downarrow & & \downarrow \hat{\Delta} \otimes \text{id} \\ \hat{A} \otimes \hat{A} & \xrightarrow{\text{id} \otimes \psi} & \hat{A} \otimes \hat{A} \otimes X \end{array}$$

is commutative. Comparing this diagram with (2.2) we see that $(\hat{A} \otimes X, \hat{\Delta} \otimes \text{id}, \psi)$ is a G -product.

Proof. Let $B = \hat{A} \otimes X$ and $\beta = \hat{\Delta} \otimes \text{id}_X$. Then $\beta \in \text{Mor}(B, \hat{A} \otimes B)$ is a left continuous action of \hat{G} on B . We shall prove that

$$(6.6) \quad B = \psi(\hat{A})(\text{I}_{\hat{A}} \otimes X).$$

Formula (6.3) means that $V_{23}U_{12}V_{23}^* = U_{12}U_{13}$. Applying to the both sides hermitian conjugation and rearranging the legs we obtain $\hat{V}_{12}^*\hat{U}_{23}\hat{V}_{12} = \hat{U}_{13}\hat{U}_{23}$. Now, using (6.4) we have:

$$(6.7) \quad (\text{id}_{\hat{A}} \otimes \psi)\hat{V} = \hat{U}_{23}\hat{V}_{12}\hat{U}_{23}^* = \hat{V}_{12}\hat{U}_{13}.$$

Therefore

$$\begin{aligned} \psi(\hat{A})(\text{I}_{\hat{A}} \otimes X) &= \left[(\omega \otimes \text{id}_{\hat{A}} \otimes \text{id}_X) \hat{V}_{12} \hat{U}_{13} (\text{I}_H \otimes \text{I}_{\hat{A}} \otimes X) : \omega \in B(H)_* \right] \\ &= \left[(\omega \otimes \text{id}_{\hat{A}} \otimes \text{id}_X) \hat{V}_{12} \hat{U}_{13} (\hat{A} \otimes \text{I}_{\hat{A}} \otimes X) : \omega \in B(H)_* \right] \\ &= \left[(\omega \otimes \text{id}_{\hat{A}} \otimes \text{id}_X) \hat{V}_{12} (\hat{A} \otimes \text{I}_{\hat{A}} \otimes X) : \omega \in B(H)_* \right] \\ &= \left[(\omega \otimes \text{id}_{\hat{A}} \otimes \text{id}_X) \hat{V}_{12} (\text{I}_H \otimes \text{I}_{\hat{A}} \otimes X) : \omega \in B(H)_* \right] = \hat{A} \otimes X \end{aligned}$$

and (6.6) follows. In the above computation ω emits \hat{A} to the left, next \hat{U}_{13} is absorbed by $\hat{A} \otimes \text{I} \otimes X$ (this is because $\hat{A} \in M(\hat{A} \otimes X)$) and finally ω absorbs \hat{A} . Using (6.6) we get: $\psi(\hat{A})B = \psi(\hat{A})\psi(\hat{A})(\text{I}_{\hat{A}} \otimes X) = \psi(\hat{A})(\text{I}_{\hat{A}} \otimes X) = B$. It shows that $\psi \in \text{Mor}(\hat{A}, B)$.

Using (6.7) we obtain

$$\begin{aligned} (\text{id}_A \otimes (\text{id}_{\hat{A}} \otimes \psi)\hat{\Delta})\hat{V} &= (\text{id}_A \otimes \text{id}_{\hat{A}} \otimes \psi)\hat{V}_{12}\hat{V}_{13} = \hat{V}_{12}\hat{V}_{13}\hat{U}_{14} \\ &= (\text{id}_A \otimes \hat{\Delta} \otimes \text{id}_X)\hat{V}_{12}\hat{U}_{13} = (\text{id}_A \otimes (\hat{\Delta} \otimes \text{id}_X)\psi)\hat{V} \end{aligned}$$

It shows that the diagram (6.5) is commutative. □

We shall describe the G -dynamical system corresponding to the G -product $(\hat{A} \otimes X, \hat{\Delta} \otimes \text{id}_X, \psi)$. Let

$$D'_1 = \left[(\text{id}_X \otimes \omega) (U(X \otimes \text{I}_H)U^*) : \omega \in B(H)_* \right]$$

and D' be the C^* -algebra generated by D'_1 . Then $D'_1, D' \subset M(X)$. For any $\omega \in A_*$ and $d' \in D'$ we set

$$\alpha'(\omega, d') = (\text{id}_X \otimes \omega) (U(d' \otimes \text{I}_H)U^*).$$

Then α' is a right weak action of G on D' and (D', α') is a G -dynamical system corresponding to the G -product $(\hat{A} \otimes X, \hat{\Delta} \otimes \text{id}_X, \psi)$. More precisely we have

Theorem 6.3. *The Landstad algebra of $(\hat{A} \otimes X, \hat{\Delta} \otimes \text{id}_X, \psi)$ coincides with $D = \text{I}_{\hat{A}} \otimes D'$. Moreover in this case the right weak action α of G on D introduced by (2.5) is related to α' by the formula*

$$(6.8) \quad \alpha(\omega, \text{I}_{\hat{A}} \otimes d') = \text{I}_{\hat{A}} \otimes \alpha'(\omega, d')$$

for any $\omega \in A_*$ and $d' \in D'$.

Proof. We use the notation introduced in the previous proof. Let $\varphi \in \text{Mor}(B, B_0(H) \otimes B)$ be the morphism introduced by (5.1). Then using (6.7) we have

$$\varphi(b) = \hat{V}_{1\psi}^*(\hat{\Delta} \otimes \text{id}_X)(b)\hat{V}_{1\psi} = \hat{U}_{13}^*\hat{V}_{12}^*(\hat{\Delta} \otimes \text{id}_X)(b)\hat{V}_{12}\hat{U}_{13} = \hat{U}_{13}^*b_{13}\hat{U}_{13} = (\hat{U}^*b\hat{U})_{13}$$

for any $b \in B$ and formula (5.2) takes the form

$$D_1 = \text{I}_{\hat{A}} \otimes \left[(\omega \otimes \text{id}_X)(\hat{U}^*B\hat{U}) : \omega \in B(H)_* \right].$$

Taking into account (6.6) and (6.4) we obtain $\hat{U}^*B\hat{U} = \hat{U}^*\psi(\hat{A})\hat{U}\hat{U}^*(\text{I}_{\hat{A}} \otimes X)\hat{U} = (\hat{A} \otimes \text{I}_X)\hat{U}^*(\text{I}_{\hat{A}} \otimes X)\hat{U}$. Inserting this result into the previous formula and absorbing \hat{A} by ω we get

$$\begin{aligned} D_1 &= \text{I}_{\hat{A}} \otimes \left[(\omega \otimes \text{id}_X)(\hat{U}^*(\text{I}_H \otimes X)\hat{U}) : \omega \in B(H)_* \right] \\ &= \text{I}_{\hat{A}} \otimes \left[(\text{id}_X \otimes \omega)(U(X \otimes \text{I}_H)U^*) : \omega \in B(H)_* \right] = \text{I}_{\hat{A}} \otimes D'_1. \end{aligned}$$

According to Proposition 5.2, the Landstad algebra of $(\hat{A} \otimes \hat{A}, \hat{\Delta} \otimes \text{id}_X, \psi)$ concides with $\text{I}_{\hat{A}} \otimes D'$. Applying to the both sides of (6.7) hermitian conjugation and rearranging the legs we obtain $V_{\psi 3} = U_{23}V_{13}$. Now, the action (2.5) takes the form

$$\begin{aligned} \alpha(\omega, \text{I}_{\hat{A}} \otimes d') &= (\text{id}_{\hat{A}} \otimes \text{id}_X \otimes \omega) (U_{23}V_{13}(\text{I}_{\hat{A}} \otimes d' \otimes \text{I}_A)V_{13}^*U_{23}^*) \\ &= \text{I}_{\hat{A}} \otimes (\text{id}_X \otimes \omega) (U(d' \otimes \text{I}_A)U^*) = \text{I}_{\hat{A}} \otimes \alpha'(\omega, d') \end{aligned}$$

for any $\omega \in A_*$ and $d' \in D'$. \square

We say that U is regular if $(\text{I}_X \otimes A)U(X \otimes \text{I}_A) = X \otimes A$. If U is regular then $D' = X$. In general $D' \neq X$.

To obtain the most obvious example of the above construction we set: $X = \hat{A}$ and $U = V$. Then $\psi = \hat{\Delta}$ and (6.5) takes the form

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\hat{\Delta}} & \hat{A} \otimes \hat{A} \\ \hat{\Delta} \downarrow & & \downarrow \hat{\Delta} \otimes \text{id} \\ \hat{A} \otimes \hat{A} & \xrightarrow{\text{id} \otimes \hat{\Delta}} & \hat{A} \otimes \hat{A} \otimes \hat{A} \end{array}$$

which is the commutative diagram expressing the coassociativity of $\hat{\Delta}$. This way for any locally compact quantum group G we have canonically associated G -product $(\hat{A} \otimes \hat{A}, \hat{\Delta} \otimes \text{id}_{\hat{A}}, \hat{\Delta})$. Let

$$(6.9) \quad D'_1 = \left[(\text{id}_{\hat{A}} \otimes \omega) \left(V(\hat{A} \otimes \text{I}_H)V^* \right) : \omega \in B(H)_* \right]$$

and D' be the C^* -algebra generated by D'_1 . Then $D'_1, D' \subset M(\hat{A})$. For any $\omega \in A_*$ and $d' \in D'$ we set

$$\alpha'(\omega, d') = (\text{id}_{\hat{A}} \otimes \omega) (V(d' \otimes \text{I}_H)V^*).$$

Then α' is a right weak action of G on D' and (D', α') is a G -dynamical system corresponding to the G -product $(\hat{A} \otimes \hat{A}, \hat{\Delta} \otimes \text{id}_{\hat{A}}, \hat{\Delta})$. If G is regular then $D' = \hat{A}$. Otherwise $D' \neq \hat{A}$.

The first known example of non-regular locally compact quantum group was quantum deformation $E_q(2)$ of the group of motions of Euclidean plane (with real deformation parameter $0 < q < 1$, see [22, 21] and [1]). Let⁴ $A = C_0(E_q(2))$ and $\Delta \in \text{Mor}(A, A \otimes A)$ be the corresponding comultiplication: $E_q(2) = (A, \Delta)$. Then $(A \otimes A, \Delta \otimes \text{id}_A, \Delta)$ is a $\widehat{E_q(2)}$ -product. Let $D = \text{I}_A \otimes D'$ be its Landstad algebra. Condition C2 says that

$$\Delta(A)(\text{I}_A \otimes D') = A \otimes A.$$

⁴notice change of notation: the role of G and \hat{G} are interchanged.

A boring calculations entering into anatomy of $E_q(2)$ shows that D' is unital. Therefore $\Delta(A) \subset A \otimes A$. For the first time this unexpected result appeared in [23].

7. A REMARK ON LANDSTAD CONDITIONS

This section is not in the main stream of the paper. It contains a generalisation of a result of Kasprzak that was used to simplify second Landstad condition (cf. formulae (3) and (4) of [5]). In this section we assume that A admits a continuous counit. (i.e: G is coameanable).

Theorem 7.1. *Let G be a coameanable locally compact quantum group, (B, β, ψ) be a G -product and $d \in M(B)$. Assume that*

$$(7.1) \quad V_{\psi 2}(d \otimes I)V_{\psi 2}^*(I \otimes a) \in M(B) \otimes A$$

for any $a \in A$. Then the following conditions are equivalent:

- (1) $\psi(\widehat{a})d \in B$ for any $\widehat{a} \in \widehat{A}$,
- (2) $d\psi(\widehat{a}') \in B$ for any $\widehat{a}' \in \widehat{A}$,
- (3) $\psi(\widehat{a})d\psi(\widehat{a}') \in B$ for any $\widehat{a}, \widehat{a}' \in \widehat{A}$.

We shall use the following result:

Proposition 7.2. *There exists a bounded net $(\widehat{e}_\lambda)_{\lambda \in \Lambda}$ of elements of \widehat{A} converging strictly to $I \in M(\widehat{A})$ such that*

$$(7.2) \quad \lim_{\lambda \in \Lambda} \|[d, \psi(\widehat{e}_\lambda)]\| = 0$$

for any $d \in M(B)$ satisfying relation (7.1). Square bracket in the above formula denotes commutator.

Proof. Let e be a counit of A and $(\omega_\lambda)_{\lambda \in \Lambda}$ be a net of normal states on A weakly converging to e :

$$\lim_{\lambda \in \Lambda} \omega_\lambda(x) = e(x)$$

for any $x \in A$. Then for any $r \in M(B) \otimes A$ we have

$$(7.3) \quad \text{norm-lim}_{\lambda \in \Lambda} (\text{id} \otimes \omega_\lambda)r = (\text{id} \otimes e)r.$$

Indeed (7.3) is obvious for $r \in M(B) \otimes_{\text{alg}} A$. Moreover the set of all $r \in M(B) \otimes A$ satisfying (7.3) is closed in norm topology. Remembering that $M(B) \otimes_{\text{alg}} A$ is norm-dense in $M(B) \otimes A$ we obtain (7.3) in full generality.

We fix an element $a \in A$ such that $e(a) = 1$ and set

$$(7.4) \quad \widehat{e}_\lambda = (\text{id} \otimes \omega_\lambda)(V^*(I \otimes a)).$$

Then $\widehat{e}_\lambda \in \widehat{A}$ for any $\lambda \in \Lambda$. With an easy calculation we obtain

$$[d, \psi(\widehat{e}_\lambda)] = (\text{id} \otimes \omega_\lambda)X,$$

where

$$\begin{aligned} X &= (d \otimes I)V_{\psi 2}^*(I \otimes a) - V_{\psi 2}^*(d \otimes a) \\ &= V_{\psi 2}^* \left[V_{\psi 2}(d \otimes I)V_{\psi 2}^*(I \otimes a) - d \otimes a \right]. \end{aligned}$$

Assume that $d \in M(B)$ satisfies (7.1). Then the expression in square bracket belongs to $M(B) \otimes A$. Therefore $X^*X \in M(B) \otimes A$. It is known that $(\text{id} \otimes e)V = I$. Therefore (e is a character) $(\text{id} \otimes e)X = d - d = 0$ and $(\text{id} \otimes e)(X^*X) = 0$. Formula (7.3) shows now that

$$\lim_{\lambda \in \Lambda} \|(\text{id} \otimes \omega_\lambda)(X^*X)\| = 0.$$

Clearly $\text{id} \otimes \omega_\lambda$ is a completely positive unital mapping. Using Kadison inequality (cf [17, Corollary 1.3.2, page 9]) we obtain

$$\begin{aligned} \|[d, \psi(\widehat{e}_\lambda)]\|^2 &= \|(\text{id} \otimes \omega_\lambda)(X^*)(\text{id} \otimes \omega_\lambda)(X)\| \\ &\leq \|(\text{id} \otimes \omega_\lambda)(X^*X)\|. \end{aligned}$$

Formula (7.2) is shown. To end the proof we have to show that the net $(\widehat{e}_\lambda)_{\lambda \in \Lambda}$ converges strictly to $I \in M(\widehat{A})$. To this end we have to choose the net $(\omega_\lambda)_{\lambda \in \Lambda}$ in a more specific way.

Let $(\omega'_\lambda)_{\lambda \in \Lambda}$ be a net of normal states on A weakly converging to e and c be an element of A such that $e(c) = 1$. For any $a \in A$ and $\lambda \in \Lambda$ we set

$$\omega_\lambda(a) = \frac{\omega'_\lambda(c^*ac)}{\omega'_\lambda(c^*c)}.$$

Then $(\omega_\lambda)_{\lambda \in \Lambda}$ is a net of normal states on A weakly converging to e . Now (7.4) takes the form

$$\widehat{e}_\lambda = \frac{(\text{id} \otimes \omega'_\lambda)((\text{I} \otimes c^*)V^*(\text{I} \otimes ca))}{\omega'_\lambda(c^*c)}$$

and

$$\widehat{a\widehat{e}_\lambda} = \frac{(\text{id} \otimes \omega'_\lambda)((\widehat{a} \otimes c^*)V^*(\text{I} \otimes ca))}{\omega'_\lambda(c^*c)},$$

$$\widehat{e}_\lambda \widehat{a} = \frac{(\text{id} \otimes \omega'_\lambda)((\text{I} \otimes c^*)V^*(\widehat{a} \otimes ca))}{\omega'_\lambda(c^*c)}$$

for any $\widehat{a} \in \widehat{A}$. The reader should notice that $(\widehat{a} \otimes c^*)V^*(\text{I} \otimes ca)$ and $(\text{I} \otimes c^*)V^*(\widehat{a} \otimes ca)$ belong to $B \otimes A$. Formula (7.3) shows now that $\widehat{a\widehat{e}_\lambda}$ and $\widehat{e}_\lambda \widehat{a}$ converge in norm to \widehat{a} . It means that \widehat{e}_λ converges strictly to $\text{I} \in M(B)$. \square

Proof of Theorem 7.1. We know that $\psi(\widehat{a}), \psi(\widehat{a}') \in M(B)$. Therefore (3) follows from (1) and from (2). We shall prove that (1) follows from (3). One can easily verify that

$$\psi(\widehat{a})d - \psi(\widehat{a})d\psi(\widehat{e}_\lambda) = \psi(\widehat{a} - \widehat{a\widehat{e}_\lambda})d - \psi(\widehat{a})[d, \psi(\widehat{e}_\lambda)].$$

Proposition 7.2 shows now that

$$\psi[\widehat{a}]d = \text{norm-}\lim_{\lambda \in \Lambda} \psi(\widehat{a})d\psi(\widehat{e}_\lambda).$$

Remembering that B is norm-closed in $M(B)$ we see that (3) implies (1). In the similar way one shows that (3) implies (2). \square

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